MATHEMATICS

RANK FUNCTIONS ON SEMIGROUPS

BY

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In this paper we will consider an abstractification of the concept of rank, as it is known for matrices in a full matrix ring over a commutative field K, for elements of some special semigroups.

S be a semigroup, that is to say a set of elements $a, b, c, \ldots$, for which an associative, not necessarily commutative multiplication is defined.

The set of all infinite sequences $\{\alpha_j\} = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of integers $\alpha_j \geqslant 0$ be denoted by V. V is partially ordered if we define $\{\alpha_j\} \leqslant \{\beta_j\}$ if $\alpha_j \leqslant \beta_j$ $(j = 1, 2, 3, \ldots)$. $\{\alpha_j\} < \{\beta_j\}$ means $\{\alpha_j\} \leqslant \{\beta_j\}$, but $\alpha_j < \beta_j$ for at least one j.

Now we define a mapping $\varrho(a)$ of S into V to be a rank function, if the following conditions are satisfied:

i. $\varrho(ab) \leqslant \varrho(a)$, $\varrho(ab) \leqslant \varrho(b)$

ii. $\varrho(a) < \varrho(b)$ implies $a = xby$ for some $x, y \in S$.

For example $\varrho(a) = (0, 0, 0, \ldots)$ for each element a of any semigroup S is a rank function.

Now we have the following lemmas:

Lemma 1. ϱ_1 and ϱ_2 being two rank functions, $\varrho_1(a) < \varrho_1(b)$ implies $\varrho_2(a) \leqslant \varrho_2(b)$.

Proof. $\varrho_1(a) < \varrho_1(b)$ implies $a = xby$ from which it follows that $\varrho_2(a) \leqslant \varrho_2(by) \leqslant \varrho_2(b)$.

An equivalence relation $\varrho_1 \sim \varrho_2$ can be defined according to the implications $\varrho_1(a) \leqslant \varrho_1(b) \leftrightarrow \varrho_2(a) \leqslant \varrho_2(b)$. Reflexivity, symmetry and transitivity of this relation are easily verified. The corresponding set of equivalence classes $\bar{\varrho}$ be denoted by $\mathfrak{A}$.

Lemma 2. $\mathfrak{A}$ is partially ordered if we define $\bar{\varrho}_1 \leqq \bar{\varrho}_2$ if $\varrho_2(a) \leqslant \varrho_2(b)$ implies $\varrho_1(a) \leqslant \varrho_1(b)$ for some $\varrho_i \in \bar{\varrho}_i$ $(i = 1, 2)$.

Proof. The definition of ordering is independent of the choice of representatives in our class $\bar{\varrho}_i$; the properties of partial ordering in $\mathfrak{A}$ are obvious.

Lemma 3. Given a countable set $\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \ldots$ of elements of $\mathfrak{A}$ there exists a (not uniquely determined) element $\bar{\varrho} \in \mathfrak{A}$ with the property $\bar{\varrho}_i \leqq \bar{\varrho}$ $(i=1, 2, 3, \ldots)$.

Proof. We use a diagonal principle

Let
$$\begin{aligned}
\varrho_1 &\in \bar{\varrho}_1 \ ; \ \varrho_1(a)=(\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots)\\
\varrho_2 &\in \bar{\varrho}_2 \ ; \ \varrho_2(a)=(\alpha_{21}, \alpha_{22}, \alpha_{23}, \ldots)\\
\varrho_3 &\in \bar{\varrho}_3 \ ; \ \varrho_3(a)=(\alpha_{31}, \alpha_{32}, \alpha_{33}, \ldots)\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

Then
$$\begin{aligned}
\varrho_1' &\sim \varrho_1 \text{ if } \varrho_1'(a)=(\alpha_{11},\ 0\ , \alpha_{12},\ 0\ ,\ 0\ , \alpha_{13}, \ldots)\\
\varrho_2' &\sim \varrho_2 \text{ if } \varrho_2'(a)=(\ 0\ , \alpha_{21},\ 0\ ,\ 0\ , \alpha_{22},\ 0\ , \ldots)\\
\varrho_3' &\sim \varrho_3 \text{ if } \varrho_3'(a)=(\ 0\ ,\ 0\ ,\ 0\ , \alpha_{31},\ 0\ ,\ 0\ , \ldots)\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Now define $\bar{\varrho}$ to be the class of ϱ where

$$\varrho(a)=(\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{31}, \alpha_{22}, \alpha_{13}, \ldots)$$

and the implication $\varrho(a) \leqslant \varrho(b) \rightarrow \varrho_i(a) \leqslant \varrho_i(b)$ is trivially true. Hence $\bar{\varrho}_i \leqq \bar{\varrho}$ $(i=1, 2, 3, \ldots)$..

Theorem 1. If $\mathfrak{A}$ itself is countable then it contains a uniquely determined element $\bar{\sigma}$ with the property $\bar{\varrho} \leqq \bar{\sigma}$ for each $\bar{\varrho} \in \mathfrak{A}$.

Proof. The existence of at least one $\bar{\sigma}$ follows at once from lemma 3. The uniqueness can be seen as follows: Suppose $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are two elements of $\mathfrak{A}$ with the mentioned property. Then $\bar{\sigma}_2 \leqq \bar{\sigma}_1$ and $\bar{\sigma}_1 \leqq \bar{\sigma}_2$ so that $\bar{\sigma}_1 = \bar{\sigma}_2$.

This class $\bar{\sigma}$ will be called rank. When S is a full matrix ring over a commutative field K the condition ii. for the rank ϱ_0 in the usual sense can be sharpened to: $\varrho_0(a) \leqslant \varrho_0(b)$ implies $a=xby$ for some elements $x, y \in S$. Hence $\varrho_0 \in \bar{\sigma}$ in this case.

From now on we shall assume that S contains an identity element e. Then $ae=ea=a$ for all $a \in S$. Now we can define an equivalence relation $a \equiv b$ $(a, b \in S)$ if both $a=xby$ and $b=taz$ hold for some $x, y, t, z \in S$. The reflexivity, symmetry and transitivity of this relation are evident, and we have as a consequence of i. $a \equiv b$ implies $\varrho(a)=\varrho(b)$ for any rank-function ϱ. Now suppose $\varrho(a)=\varrho(b)$, where $a \not\equiv b$. Then we can prove

Lemma 4. If $\varrho(a)=\varrho(b)$ and $a \not\equiv b$, then there exists a class $\bar{\varrho}'$ such that $\bar{\varrho} \leqq \bar{\varrho}'$, $\bar{\varrho} \neq \bar{\varrho}'$.

Proof. We have two cases

a. None of the equations $a=xby$ and $b=taz$ admits a solution.

b. One of the mentioned equations, say $a=xby$ holds for some $x, y \in S$.

Suppose $\qquad \varrho(c) = (\gamma_1, \gamma_2, \gamma_3, \ldots) \qquad$ for $c \in S$

$$\varrho(a) = \varrho(b) = (\alpha_1, \alpha_2, \alpha_3, \ldots).$$

In the successive cases we define ϱ'

a.
$$\begin{aligned}
\varrho'(c) &= (0, 0, \gamma_1, \gamma_2, \gamma_3, \ldots) \text{ for } a|c \text{ or } b|c,\ c \not\equiv a, b \\
\varrho'(c) &= (1, 0, \alpha_1, \alpha_2, \alpha_3, \ldots) \text{ for } c \equiv a \\
\varrho'(c) &= (0, 1, \alpha_1, \alpha_2, \alpha_3, \ldots) \text{ for } c \equiv b \\
\varrho'(c) &= (1, 1, \gamma_1, \gamma_2, \gamma_3, \ldots) \text{ otherwise,}
\end{aligned}$$

$a|c$ meaning $c = xay$ for some $x, y \in S$.

b.
$$\begin{aligned}
\varrho'(c) &= (0, \gamma_1, \gamma_2, \gamma_3, \ldots) \text{ for } a|c,\ c \not\equiv a \\
\varrho'(c) &= (0, \alpha_1, \alpha_2, \alpha_3, \ldots) \text{ for } c \equiv a \\
\varrho'(c) &= (1, \gamma_1, \gamma_2, \gamma_3, \ldots) \text{ otherwise.}
\end{aligned}$$

If now ϱ is chosen within the class $\bar{\sigma}$ such a class $\bar{\varrho}'$ does not exist, so we have proved

Theorem 2. $a \equiv b \leftrightarrow \varrho(a) = \varrho(b)$ for $\varrho \in \bar{\sigma}$.

If at last S is supposed to be a commutative semigroup with an identity element the classes $\widetilde{a}, \widetilde{b}$ of the equivalence relation $\equiv$ can be multiplied in a meaningful way by the definition $\widetilde{a} \cdot \widetilde{b} = \widetilde{ab}$, and form therefore a semigroup $\widetilde{S}$, on which any rank function ϱ on S induces another rank function $\widetilde{\varrho}$, by the relation $\widetilde{\varrho}(\widetilde{a}) = \varrho(a)$. According to theorem 2 the rank, defined on S induces a class of discrete rank functions on $\widetilde{S}$, which is obviously the rank of $\widetilde{S}$.

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